

# Multiparameter Quantum Function Algebra at Roots of 1

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*Mathematics Subject Classification (1991)* : 16W30, 17B37, 81R50

## Introduction

In this paper we consider a multiparameter deformation  $F_q^\varphi[G]$  of the quantum function algebra associated to a simple algebraic group  $G$ . This deformation has been introduced by Reshetikhin ([R], cf. also [D-K-P1]) and is constructed from a skew endomorphism  $\varphi$  of the weight lattice of  $G$ . When  $\varphi$  is zero we get the standard quantum group, that is the algebra studied by [H-L1-2-3], [Jo] and, in the compact case, by [L-S2]. In the case ( $\varphi = 0$ ) and when the quantum parameter  $q$  is a root of unity, important results are contained in [D-L] and [D-P2]. A general  $\varphi$  has been considered in [L-S2] for  $G$  compact and  $q$  generic. Here we study the representation theory at roots of one for a non trivial  $\varphi$ .

Our arguments are similar to those used by De Concini and Lyubashenko. Nevertheless there is a substantial difference: when  $\varphi = 0$  the major tool is to understand in detail the  $SL(2)$ -case which allows to construct representations. Unfortunately there are not multiparameter deformations for  $SL(2)$ . Moreover the usual right and left actions of the braid group on  $F_q^\varphi[G]$  are not so powerful as in the case  $\varphi = 0$ .

We first (sections **1.,2.,3.**) give some properties of the multiparameter quantum function algebra  $F_\varepsilon^\varphi[G]$  at  $\varepsilon$ ,  $l$ -th root of 1. To do this we principally use a duality, given in [C-V], between some Borel type subHopfalgebras of  $F_q^\varphi[G]$ . In section **4.** we compute the dimension of the symplectic leaves of  $G$  for the Poisson structure determined by  $\varphi$ . Our main result (cor. **5.7**) is the link between this dimension and the dimension of the representations of  $F_\varepsilon^\varphi[G]$ , for "good"  $l$ . More precisely, we can see  $F_\varepsilon^\varphi[G]$  as a bundle of algebras on  $G$ . Its theory of representations is constant over the  $T$ -biinvariant Poisson submanifold of  $G$  ( $T$  being the Cartan torus of  $G$ ) and we have

**Theorem** *Let  $l$  be a "good" integer (see **5.5**) and let  $p$  be a point in the symplectic leaf  $\Theta$  of  $G$ . Then the dimension of any representation of  $F_\varepsilon^\varphi[G]$  lying over  $p$  is divisible by  $l^{\frac{1}{2} \dim \Theta}$ .*

Finally, using the results in the first three sections, we describe explicitly (**5.8**) a class of representations of  $F_\varepsilon^\varphi[G]$ .

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**Notations.** For the comultiplication in a coalgebra we use the notation  $\Delta x = x_{(1)} \otimes x_{(2)}$ . If  $H$  is a Hopf algebra, we denote by  $H^{op}$  the same coalgebra with the opposite multiplication and by  $H_{op}$  the same algebra with the opposite comultiplication.

Let  $F$  be a field and let  $(H_i, m_i, \eta_i, \Delta_i, \varepsilon_i, S_i)$ ,  $i = 1, 2$ , be Hopf algebras. Then an  $F$ -linear pairing  $\pi : H_1 \otimes H_2 \rightarrow F$  is called an *Hopf algebra pairing* [T] if :

$$\pi(uv \otimes h) = \pi(u \otimes h_{(1)})\pi(v \otimes h_{(2)}), \quad \pi(u \otimes hl) = \pi(u_{(1)} \otimes h)\pi(u_{(2)} \otimes l)$$

$$\pi(\eta_1 1 \otimes h) = \varepsilon_2 h, \quad \pi(u \otimes \eta_2 1) = \varepsilon_1 u$$

$$\pi(S_1 u \otimes h) = \pi(u \otimes S_2 h),$$

for  $u, v \in H_1$ ,  $h, l \in H_2$ . Moreover  $\pi$  is said *perfect* if it is not degenerate.

We denote by  $R$  the ring  $\mathbb{Q}[q, q^{-1}]$  and by  $K$  its quotient field  $\mathbb{Q}(q)$ . Take a positive integer  $l$  and let  $p_l(q)$  be the  $l$ th cyclotomic polynomial. We define  $\mathbb{Q}(\varepsilon) = \mathbb{Q}[q]/(p_l(q))$  ( $\varepsilon$  being a primitive  $l$ th root of unity). Finally we recall the definition of the  $q$ -numbers:

$$(n)_q = \frac{q^n - 1}{q - 1}, \quad (n)_q! = \prod_{m=1}^n (m)_q, \quad \binom{n}{m}_q = \frac{(n)_q!}{(m)_q! (n-m)_q!},$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{m=1}^n [m]_q, \quad \left[ \begin{matrix} n \\ m \end{matrix} \right]_q = \frac{[n]_q!}{[m]_q! [n-m]_q!}.$$

### 1. The Multiparameter Quantum Group

**1.1.** Let  $A = (a_{ij})$  be an indecomposable  $n \times n$  Cartan matrix; that is let  $a_{ij}$  be integers with  $a_{ii} = 2$  and  $a_{ij} \leq 0$  for  $i \neq j$  and let  $(d_1, \dots, d_n)$  be a fixed  $n$ -uple of relatively prime positive integers  $d_i$  such that the matrix  $TA$  is symmetric and positive definite. Here  $T$  is the diagonal matrix with entries  $d_i$ .

Consider the free abelian group  $P = \sum_{i=1}^n \mathbb{Z}\omega_i$  with basis  $\{\omega_i | i = 1, \dots, n\}$  and define

$$\alpha_i = \sum_{j=1}^n a_{ji}\omega_j \quad (i = 1, \dots, n), \quad Q = \sum_{i=1}^n \mathbb{Z}\alpha_i, \quad P_+ = \sum_{i=1}^n \mathbb{Z}_+\omega_i;$$

$P$  and  $Q$  are called respectively the weight and the root lattice, the elements of  $P_+$  are the dominant weights.

Define a bilinear  $\mathbb{Z}$ -valued pairing on  $P \times Q$  by the rule  $(\omega_i, \alpha_j) = d_i \delta_{ij}$  ( $\delta_{ij}$  is the Kronecher symbol); it can be extended to symmetric pairings

$$P \times P \rightarrow \mathbb{Z}\left[\frac{1}{\det(A)}\right], \quad QP \times QP \rightarrow \mathbb{Q}$$

where  $QP = \sum_{i=1}^n \mathbb{Q}\omega_i$ .

To this setting is associated a complex simple finite dimensional Lie algebra  $\mathfrak{g}$  and a complex connected simply connected simple algebraic group  $G$ .

**1.2.** Fix an endomorphism  $\varphi$  of the  $\mathbb{Q}$ -vector space  $\mathbb{Q}P$  which satisfies the following conditions :

$$(1.1) \quad \begin{cases} (\varphi x, y) = -(x, \varphi y) & \forall x, y \in \mathbb{Q}P \\ \varphi \alpha_i = \delta_i = 2\tau_i & \tau_i \in \mathbb{Q}, i = 1, \dots, n \\ \frac{1}{2}(\varphi \lambda, \mu) \in \mathbb{Z} & \forall \lambda, \mu \in P \end{cases}$$

We will see later the motivation of the third assumption, now observe that it implies  $\varphi P \subseteq P$ .

If  $\tau_i = \sum_{j=1}^n x_{ji} \omega_j = \sum_{j=1}^n y_{ji} \alpha_j$ , let put  $X = (x_{ij})$ ,  $Y = (y_{ij})$ . Then  $TX$  is an antisymmetric matrix and the last two conditions in (1.1) are equivalent to the following :

$$Y \in M_n(\mathbb{Z}) \cap T^{-1}A_n(\mathbb{Z})A$$

where  $A_n(\mathbb{Z})$  denotes the submodule of  $M_n(\mathbb{Z})$  given by the antisymmetric matrices.

The maps

$$1 \pm \varphi : \mathbb{Q}P \longrightarrow \mathbb{Q}P, \alpha_i \mapsto \alpha_i \pm \delta_i$$

are  $\mathbb{Q}$ -isomorphisms (cf. [C-V]); moreover we have

$$((1 + \varphi)^{\pm 1} \lambda, \mu) = (\lambda, (1 - \varphi)^{\pm 1} \mu), \forall \lambda, \mu \in P$$

and so  $(1 + \varphi)^{\pm 1} (1 - \varphi)^{\mp 1}$  are isometries of  $\mathbb{Q}P$ . Let us put  $r = (1 + \varphi)^{-1}$ ,  $\bar{r} = (1 - \varphi)^{-1}$ .

We like to stress that if we want to enlarge the results of this paper to the semisimple case it is enough to ask that  $2AYA^{-1} \in M_n(\mathbb{Z})$ , which guarantees  $\varphi P \subseteq P$ .

**1.3.** The *multiparameter simply connected quantum group*  $U_q^\varphi(\mathfrak{g})$  associated to  $\varphi$  ([R],[D-K-P1],cf. [C-V]) is the  $K$ -algebra on generators  $E_i, F_i, K_{\omega_i}^{\pm 1}$ , ( $i = 1, \dots, n$ ) with the same relations of the Drinfel'd-Jimbo quantum group  $U_q(\mathfrak{g}) = U_q^0(\mathfrak{g})$  and with an Hopf algebra structure given by the following comultiplication  $\Delta_\varphi$ , counity  $\epsilon_\varphi$  and antipode  $S_\varphi$  defined on generators ( $i = 1, \dots, n$ ;  $\lambda \in P$ )

$$\begin{cases} \Delta_\varphi E_i = E_i \otimes K_{-\tau_i} + K_{-\alpha_i + \tau_i} \otimes E_i \\ \Delta_\varphi F_i = F_i \otimes K_{\alpha_i + \tau_i} + K_{-\tau_i} \otimes F_i \\ \Delta_\varphi K_\lambda = K_\lambda \otimes K_\lambda \end{cases}, \quad \begin{cases} \epsilon_\varphi E_i = 0 \\ \epsilon_\varphi F_i = 0 \\ \epsilon_\varphi K_\lambda = 1 \end{cases}, \quad \begin{cases} S_\varphi E_i = -K_{\alpha_i} E_i \\ S_\varphi F_i = -F_i K_{-\alpha_i} \\ S_\varphi K_\lambda = K_{-\lambda} \end{cases},$$

where for  $\lambda = \sum_{i=1}^n m_i \omega_i \in P$  we use the notation  $K_\lambda = \prod_{i=1}^n K_{\omega_i}^{m_i}$ .

Put  $K_i = K_{\alpha_i}$ ,  $q_i = q^{d_i}$ ; we recall the relations in the algebra  $U_q(\mathfrak{g})$  ([D1],[J]) :

$$(1.2) \quad K_{\omega_i} K_{\omega_i}^{-1} = 1 = K_{\omega_i}^{-1} K_{\omega_i}, \quad K_{\omega_i} K_{\omega_j} = K_{\omega_j} K_{\omega_i},$$

$$(1.3) \quad K_{\omega_i} E_j K_{\omega_i}^{-1} = q_i^{\delta_{ij}} E_j, \quad K_{\omega_i} F_j K_{\omega_i}^{-1} = q_i^{-\delta_{ij}} F_j,$$

$$(1.4) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$(1.5) \quad \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} G_i^{1-a_{ij}-m} G_j G_i^m = 0 \quad (i \neq j),$$

in the two cases  $G_i = E_i, F_i$ .

**1.4.** Let  $U_q^\varphi(\mathfrak{b}_+)$  and  $\overline{U}_q^\varphi(\mathfrak{b}_+)$  be the sub-Hopf algebras of  $U_q^\varphi(\mathfrak{g})$  generated by the  $E'_i$ 's ( $i = 1, \dots, n$ ) and respectively by the sets  $\{K_\lambda | \lambda \in P\}$ ,  $\{K_\lambda | \lambda \in Q\}$ . Similarly let  $U_q^\varphi(\mathfrak{b}_-)$  and  $\overline{U}_q^\varphi(\mathfrak{b}_-)$  be the sub-Hopf algebras of  $U_q^\varphi(\mathfrak{g})$  generated by the  $F'_i$ 's ( $i = 1, \dots, n$ ) and respectively by  $P$  and  $Q$  in the multiplicatively notation of the  $K'_\lambda$ 's.

Take an element  $u$  in the algebraic closure of  $K$  such that  $u^{det(A+D)} = q$ . Then we know (cf. [C-V]) that the following bilinear map is a perfect Hopf algebra pairing :

$$\pi_\varphi : U_q^\varphi(\mathfrak{b}_-)_{op} \otimes U_q^\varphi(\mathfrak{b}_+) \longrightarrow \mathbb{Q}(u), \quad \begin{cases} \pi_\varphi(K_\lambda, K_\mu) = q^{(r(\lambda), \mu)} \\ \pi_\varphi(K_\lambda, E_i) = \pi_\varphi(F_i, K_\lambda) = 0, \\ \pi_\varphi(E_i, F_j) = \frac{\delta_{ij}}{q_i - q_i^{-1}} q^{(r(\tau_i), \tau_i)} \end{cases},$$

for  $\lambda, \mu \in P$ ,  $i = 1, \dots, n$ . Consider now the antisomorphism  $\zeta_\varphi$  of Hopf algebras relative to a  $\mathbb{Q}$ -algebra antisomorphism  $\zeta : K \longrightarrow K$ ,  $q \mapsto q^{-1}$  of the basic ring, namely

$$\zeta_\varphi : U_q^\varphi(\mathfrak{g}) \mapsto U_q^{-\varphi}(\mathfrak{g}), \quad E_i \mapsto F_i, \quad F_i \mapsto E_i, \quad K_\lambda \mapsto K_{-\lambda},$$

which send  $U_q^\varphi(\mathfrak{b}_+)$  into  $U_q^{-\varphi}(\mathfrak{b}_-)$  and viceversa. Then  $\overline{\pi}_\varphi = \zeta \circ \pi_{-\varphi} \circ (\zeta_\varphi \otimes \zeta_\varphi)$  is a perfect Hopf algebra pairing :

$$\overline{\pi}_\varphi : U_q^\varphi(\mathfrak{b}_+)_{op} \otimes U_q^\varphi(\mathfrak{b}_-) \longrightarrow \mathbb{Q}(u), \quad \begin{cases} \overline{\pi}_\varphi(K_\lambda, K_\mu) = q^{-(\overline{r}(\lambda), \mu)} \\ \overline{\pi}_\varphi(E_i, K_\lambda) = \overline{\pi}_\varphi(K_\lambda, F_i) = 0, \\ \overline{\pi}_\varphi(E_i, F_j) = \frac{\delta_{ij}}{q_i^{-1} - q_i} q^{-(\overline{r}(\tau_i), \tau_i)} \end{cases}.$$

For a monomial  $E_{\underline{i}} = E_{i_1} \cdots E_{i_r}$  in the  $E'_i$ 's and a monomial  $F_{\underline{i}} = F_{i_1} \cdots F_{i_r}$  in the  $F'_i$ 's we define the weight  $p(E_{\underline{i}})$  and  $p(F_{\underline{i}})$  in the following way :

$$(1.6) \quad p(E_r) = p(F_r) = \alpha_r, \quad p(E_{\underline{i}}) = p(F_{\underline{i}}) = \sum_{j=1}^n \alpha_{i_j}.$$

If  $v$  is a monomial in the  $E_i$ 's or in the  $F_i$ 's with  $p(v) = \varepsilon$  we will write  $s(v)$ ,  $r(v)$ ,  $\overline{r}(v)$ , instead of  $\frac{1}{2}\varphi(\varepsilon)$ ,  $\frac{1}{2}r(\varphi(\varepsilon))$ ,  $\frac{1}{2}\overline{r}(\varphi(\varepsilon))$ .

The next lemma is proved in [C-V].

**1.5. Lemma** *For  $x$  and  $y$  homogeneous polynomials in the  $E_i$ 's and the  $F_i$ 's respectively and for  $\lambda, \mu \in P$  it holds :*

$$(i) \quad \pi_\varphi(yK_\lambda, xK_\mu) = \pi_\varphi(y, x)q^{(r(\lambda), \mu - s(x)) - (r(y), \mu)} = \pi_0(y, x)q^{(r(\lambda) - r(y), \mu - s(x))},$$

$$(ii) \quad \overline{\pi}_\varphi(K_\lambda x, K_\mu y) = \overline{\pi}_\varphi(x, y)q^{(\overline{r}(\lambda), s(x) - \mu) + (\overline{r}(y), \mu)} = \overline{\pi}_0(x, y)q^{(\overline{r}(\lambda) - \overline{r}(y), s(x) - \mu)}.$$

**1.6.** Consider the following  $R$ -subalgebra of  $U_q^\varphi(\mathfrak{g})$  :

$$\Gamma(\mathfrak{t}) = \{f \in K[\mathbb{Q}] \mid \pi_\varphi(f, K_{(1-\varphi)\lambda}) = \pi_0(f, K_\lambda) \in R \ \forall \lambda \in P\}.$$

In [D-L] is given a  $K$ -basis  $\{\xi_t \mid t = (t_1, \dots, t_n) \in \mathbb{Z}_+^n\}$  of  $K[\mathbb{Q}]$  which is an  $R$ -basis of  $\Gamma(\mathfrak{t})$ , namely

$$\xi_t = \prod_{i=1}^n \binom{K_i; 0}{t_i} K_i^{-[\frac{t_i}{2}]}; \quad \binom{K_i; 0}{t_i} = \prod_{s=1}^t \frac{K_i q_i^{-s+1} - 1}{q_i^s - 1}$$

(for a positive integer  $s$ ,  $[s]$  denote the integer part). Note that

$$(1.7) \quad \{f \in K[(1+\varphi)P] \mid \pi_\varphi(f \otimes \Gamma(\mathfrak{t})) \subseteq R\} = R[(1+\varphi)P],$$

$$(1.8) \quad \{f \in K[(1-\varphi)P] \mid \pi_\varphi(\Gamma(\mathfrak{t}) \otimes f) \subseteq R\} = R[(1-\varphi)P].$$

**1.7.** Let  $W$  be the Weyl group associated to the Cartan matrix  $A$ , that is let  $W$  be the finite subgroup of  $GL(P)$  generated by the automorphisms  $s_i$  of  $P$  given by  $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ . If  $\Omega = \{\alpha_1, \dots, \alpha_n\}$ , the root system corresponding to  $A$  is  $\Phi = W\Omega$  while the set of positive roots is  $\Phi_+ = \Phi \cap \sum_{i=1}^n \mathbb{Z}_+ \alpha_i$ . Fix a reduced expression for the longest element  $\omega_0$  of  $W$ , say  $\omega_0 = s_{i_1} \cdots s_{i_N}$  and consider the usual total ordering on the set  $\Phi_+$  induced by this choice :

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \dots, \beta_N = s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N}.$$

Introduced, for  $k = 1, \dots, N$ , the corresponding root vectors :

$$G_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(G_{i_k}), \quad G_i = E_i, F_i,$$

where the  $T_i$ 's are the algebra automorphisms of  $U_q(\mathfrak{g})$  (and so of  $U_q^\varphi(\mathfrak{g})$ ) introduced by Lusztig up to change  $q \leftrightarrow q^{-1}$ ,  $K_\lambda \leftrightarrow K_{-\lambda}$  (see[L2]).

For a positive integer  $s$  define

$$G_i^{(s)} = \frac{G_i^s}{[s]_{q_i}!}, \quad G_{\beta_k}^{(s)} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(G_{i_k}^{(s)}),$$

always in the two cases  $G_i = E_i, F_i$ .

For  $\alpha \in \Phi_+$  let put

$$\begin{aligned} q_\alpha &= q^{\frac{(\alpha, \alpha)}{2}}, \quad \tau_\alpha = \frac{1}{2} \varphi \alpha; \\ e_\alpha^\varphi &= (q_\alpha^{-1} - q_\alpha) E_\alpha K_{\tau_\alpha}, \quad f_\alpha^\varphi = (q_\alpha - q_\alpha^{-1}) F_\alpha K_{\tau_\alpha}; \\ e_i^\varphi &= e_{\alpha_i}^\varphi, \quad f_i^\varphi = f_{\alpha_i}^\varphi. \end{aligned}$$

Note that  $K_{\tau_\alpha}$  commutes with every monomial of weight  $\alpha$ .

**1.8.** Define  $R_q^\varphi[B_-]'$  and  $R_q^\varphi[B_-]''$  as the  $R$ -subalgebras of  $U_q^\varphi(\mathfrak{b}_+)^{op}$  and  $U_q^\varphi(\mathfrak{b}_+)_{op}$  respectively generated by the elements  $e_\alpha^\varphi$ ,  $K_{(1-\varphi)\omega_i}$  ( $i = 1, \dots, n$ ,  $\alpha \in \Phi_+$ ). Similarly denote by  $R_q^\varphi[B_+]'$  and  $R_q^\varphi[B_+]''$  the  $R$ -subalgebras of  $U_q^\varphi(\mathfrak{b}_-)^{op}$  and  $U_q^\varphi(\mathfrak{b}_-)_{op}$  respectively generated by the elements  $f_\alpha^\varphi$ ,  $K_{(1+\varphi)\omega_i}$  ( $i = 1, \dots, n$ ,  $\alpha \in \Phi_+$ ). Then, by restriction from  $\pi_\varphi$ , we obtain the following two pairings :

$$\pi'_\varphi : \overline{U}_q^\varphi(\mathfrak{b}_-) \otimes_R R_q^\varphi[B_-]' \longrightarrow K \quad \pi''_\varphi : R_q^\varphi[B_+]'' \otimes_R \overline{U}_q^\varphi(\mathfrak{b}_+) \longrightarrow K$$

while by restriction from  $\overline{\pi}_\varphi$  we get the other two :

$$\overline{\pi}'_\varphi : \overline{U}_q^\varphi(\mathfrak{b}_+) \otimes_R R_q^\varphi[B_+]' \longrightarrow K \quad \overline{\pi}''_\varphi : R_q^\varphi[B_-]'' \otimes_R \overline{U}_q^\varphi(\mathfrak{b}_-) \longrightarrow K.$$

We get :

$$\begin{cases} \pi'_\varphi(F_j, e_i^\varphi) = -\delta_{ij} \\ \pi'_\varphi(K_\lambda, K_{(1-\varphi)\mu}) = q^{(\lambda, \mu)} \end{cases} \quad \begin{cases} \pi''_\varphi(f_i^\varphi, E_j) = \delta_{ij} \\ \pi''_\varphi(K_{(1+\varphi)\mu}, K_\lambda) = q^{(\mu, \lambda)} \end{cases}$$

$$\begin{cases} \overline{\pi}'_\varphi(E_j, f_i^\varphi) = -\delta_{ij} \\ \overline{\pi}'_\varphi(K_\lambda, K_{(1+\varphi)\mu}) = q^{-(\lambda, \mu)} \end{cases} \quad \begin{cases} \overline{\pi}''_\varphi(e_i^\varphi, F_j) = \delta_{ij} \\ \overline{\pi}''_\varphi(K_{(1-\varphi)\mu}, K_\lambda) = q^{-(\mu, \lambda)} \end{cases}.$$

We can choose as bases of  $U_q^\varphi(\mathfrak{b}_+)$  and  $U_q^\varphi(\mathfrak{b}_-)$  the elements (see [L2],[D-L]):

$$\xi_{m,t} = \prod_{j=N}^1 E_{\beta_j}^{(m_j)} \prod_{i=1}^n \binom{K_i; 0}{t_i} K_i^{-[\frac{t_i}{2}]}, \quad \eta_{m,t} = \prod_{j=N}^1 F_{\beta_j}^{(m_j)} \prod_{i=1}^n \binom{K_i; 0}{t_i} K_i^{-[\frac{t_i}{2}]}$$

### 1.9. Proposition

$$q^{-\sum_{i < j} (n_i \tau_i, n_j \beta_j)} \pi'_\varphi(\eta_{m,t}, \prod_{j=N}^1 (e_{\beta_j}^\varphi)^{m_j} K_{(1-\varphi)\lambda}) = q^{\sum_{i < j} (n_i \tau_i, n_j \tau_j)} \pi''_\varphi(\prod_{j=N}^1 (f_{\beta_j}^\varphi)^{n_j} K_{(1+\varphi)\lambda}, \xi_{m,t}) =$$

$$\left( \prod_{i=1}^N \delta_{n_i, m_i} q_{\beta_i}^{-\frac{n_i(n_i-1)}{2}} \prod_{i=1}^N \binom{(\alpha_i, \lambda)}{t_i} \right)_q q^{-(\alpha_i, \lambda)[\frac{t_i}{2}]} q^{-\sum_{i=1}^N (n_i \tau_i, \lambda)}.$$

Similar formulas hold for  $\overline{\pi}'_\varphi$  and  $\overline{\pi}''_\varphi$ .

*Proof.* First of all observe that

$$\Delta_\varphi e_i^\varphi = e_i^\varphi \otimes 1 + K_{-(1-\varphi)\alpha_i} \otimes e_i^\varphi, \quad \Delta_\varphi f_i^\varphi = f_i^\varphi \otimes K_{(1+\varphi)\alpha_i} + 1 \otimes f_i^\varphi,$$

and, for  $\alpha \in \Phi_+$ ,

$$\Delta_\varphi e_\alpha^\varphi = e_\alpha^\varphi \otimes 1 + K_{-(1-\varphi)\alpha} \otimes e_\alpha^\varphi + e, \quad \Delta_\varphi f_\alpha^\varphi = f_\alpha^\varphi \otimes K_{(1+\varphi)\alpha} + 1 \otimes f_\alpha^\varphi + f,$$

where  $e$  ( $f$ ) is a sum of terms  $u_i \otimes v_i$ ,  $u_i$  and  $v_i$  being linear combination of monomials in the  $e_\beta^\varphi$  ( $f_\beta^\varphi$ ) and  $K_\lambda$  and  $ht(\beta) < ht(\alpha)$ . Moreover

$$\pi'_\varphi(F_\alpha, e_\alpha^\varphi) = \pi'_\varphi(F_\alpha, e_\alpha K_{\tau_\alpha}) = \pi_0(F_\alpha, e_\alpha) \quad \forall \alpha \in \Phi_+.$$

Put now  $F = \eta_{n,0}$ ,  $M = \eta_{0,t}$ ,  $e^\varphi = \prod_{j=1}^N (e_{\beta_j}^\varphi)^{m_j}$ , then

$$\begin{aligned}\pi'_\varphi(FM, e^\varphi K_{(1-\varphi)\lambda}) &= \pi'_\varphi(F \otimes M, \Delta_\varphi(e^\varphi K_{(1-\varphi)\lambda})) = \pi'_\varphi(F, e^\varphi K_{(1-\varphi)\lambda}) \pi'_\varphi(M, K_{(1-\varphi)\lambda}) = \\ &= \pi'_\varphi(\Delta_\varphi F, K_{(1-\varphi)\lambda} \otimes e^\varphi) \pi'_0(M, K_\lambda) = q^{-(r(F), (1-\varphi)\lambda)} \pi'_\varphi(F, e^\varphi) \pi'_0(M, K_\lambda) = \\ &= q^{-(s(F), \lambda)} \pi'_\varphi(F, e^\varphi) \pi'_0(M, K_\lambda).\end{aligned}$$

Now, if  $e^0 = \prod_{j=1}^N (e_{\beta_j}^0)^{m_j}$  we have

$$\pi'_\varphi(F, e^\varphi) = q^{\sum_{i < j} (m_i \tau_i, m_j \tau_j)} \pi'_\varphi(F, e^0 K_{\sum_i m_i \tau_{\beta_i}}) = q^{\sum_{i < j} (m_i \tau_i, m_j \tau_j)} \pi'_0(F, e^0),$$

where the powers of  $q$  arises from the commutation of  $K_{\tau_\alpha}$  and the last equality from 1.5. Since the value of  $\pi'_0(F, e^0)$  is calculated in [D-L] (formula (3.2)) we are done.

For the other equality as well as for the case of  $\bar{\pi}_\varphi$  we proceed in the same way.  $\square$

**1.10.** Define the following  $R$ -submodules of  $U_q^\varphi(\mathfrak{g})$  :

$$\begin{aligned}\Gamma^\varphi(\mathfrak{b}_+) &= \{x \in \overline{U}_q^\varphi(\mathfrak{b}_+) \mid \pi''_\varphi(R_q^\varphi[B_+]'' \otimes x) \subset R\} \\ \Gamma^\varphi(\mathfrak{b}_-) &= \{x \in \overline{U}_q^\varphi(\mathfrak{b}_-) \mid \pi'_\varphi(x \otimes R_q^\varphi[B_-]'^{op}) \subset R\}.\end{aligned}$$

It is clear from prop.1.2. that the  $\xi_{m,t}$ 's and the  $\eta_{m,t}$ 's are  $R$ -bases of  $\Gamma^\varphi(\mathfrak{b}_+)$  and  $\Gamma^\varphi(\mathfrak{b}_-)$  respectively and so first of all they are algebras (cf. [L]) and secondly as algebras they are isomorphic to  $\Gamma^0(\mathfrak{b}_+)$  and  $\Gamma^0(\mathfrak{b}_-)$  respectively. They are also sub-coalgebras of  $U_q^\varphi(\mathfrak{g})$ , namely

$$(1.9) \quad \begin{cases} \Delta_\varphi E_i^{(p)} = \sum_{r+s=n} q_i^{-rs} E_i^{(r)} K_{s(\tau_i - \alpha_i)} \otimes E_i^{(s)} K_{-r\tau_i} \\ \Delta_\varphi F_i^{(p)} = \sum_{r+s=n} q_i^{-rs} F_i^{(r)} K_{-s\tau_i} \otimes F_i^{(s)} K_{r(\alpha_i + \tau_i)} , \\ \Delta_\varphi \begin{pmatrix} K_i; 0 \\ t \end{pmatrix} = \sum_{r+s=t} q_i^{-rs} \begin{pmatrix} K_i; 0 \\ t \end{pmatrix} \otimes \begin{pmatrix} K_i; 0 \\ t \end{pmatrix} \end{cases}$$

(the first two equalities are proved in [C-V], the last in [D-L]).

**1.11.** As a consequence of 1.9. we get, by restriction, two pairings

$$\pi'_\varphi : \Gamma^\varphi(\mathfrak{b}_-) \otimes_R R_q^\varphi[B_-]' \longrightarrow R, \quad \pi''_\varphi : R_q^\varphi[B_+]'' \otimes_R \Gamma^\varphi(\mathfrak{b}_+) \longrightarrow R.$$

Moreover the same formulas in 1.9. and (1.7), (1.8) give

$$\begin{aligned}\{f \in U_q^\varphi(\mathfrak{b}_+) \mid \pi'_\varphi(\Gamma^\varphi(\mathfrak{b}_-)_{op} \otimes f) \subset R\} &= R_q^\varphi[B_-]', \\ \{f \in U_q^\varphi(\mathfrak{b}_-) \mid \pi''_\varphi(f \otimes \Gamma^\varphi(\mathfrak{b}_+)^{op}) \subset R\} &= R_q^\varphi[B_+]''.\end{aligned}$$

Clearly analogous results hold for  $\bar{\pi}'_\varphi$ ,  $\bar{\pi}''_\varphi$  and so we have the two perfect pairings

$$\bar{\pi}'_\varphi : \Gamma^\varphi(\mathfrak{b}_+) \otimes_R R_q^\varphi[B_+]'' \longrightarrow R, \quad \bar{\pi}''_\varphi : R_q^\varphi[B_-]' \otimes_R \Gamma^\varphi(\mathfrak{b}_-) \longrightarrow R.$$

Most of the definitions and notations introduced up to now are generalisations to the multiparameter case of the ones given in [D-L]. In extending De Concini-Lyubashenko results we shall only write the parts of the proofs which differ from theirs.

**1.12. Lemma** *The algebras  $R_q^\varphi[B_-]'$ ,  $R_q^\varphi[B_+]$ ,  $R_q^\varphi[B_+]''$ ,  $R_q^\varphi[B_-]''$  have an Hopf algebra structure for which  $\pi'_\varphi$ ,  $\bar{\pi}'_\varphi$ ,  $\pi''_\varphi$ ,  $\bar{\pi}''_\varphi$  become perfect Hopf algebra pairings.*

*Proof.* Consider for example  $R_q^\varphi[B_-]'$  and let  $U_+$  be the sub-K-algebra of  $U_q^\varphi(\mathfrak{b}_+)^{op}$  generated by  $\{e_\alpha^\varphi, K_{(1-\varphi)\lambda} \mid \alpha \in \Phi_+, \lambda \in P\}$ . We know that (see [L]) the set  $\{e_i^\varphi, K_{(1-\varphi)\lambda} \mid i = 1, \dots, n, \lambda \in P\}$  is a generating set for  $U_+$ . Moreover since

$$\Delta_\varphi e_i^\varphi = e_i^\varphi \otimes 1 + K_{-(1-\varphi)\alpha_i} \otimes e_i^\varphi, \quad S_\varphi e_i^\varphi = -K_{-(1-\varphi)\alpha_i} e_i^\varphi, \quad \varepsilon_\varphi e_i^\varphi = 0,$$

$U_+$  is an Hopf algebra. So  $\Delta_\varphi e \in U_+ \otimes U_+$  for every  $e \in R_q^\varphi[B_-]'$ . In order to see that indeed  $\Delta_\varphi e \in R_q^\varphi[B_-] \otimes R_q^\varphi[B_-]$  and to conclude the proof we can proceed as in [D-L](Lemma 3.4).  $\square$

## 2. The Multiparameter Quantum Function Algebra

**2.1.** Consider the full subcategory  $\mathcal{C}_\varphi$  in  $U_q^\varphi(\mathfrak{g}) - mod$  consisting of all finite dimensional modules on which the  $K_i$ 's act as powers of  $q$ . If  $V$  and  $W$  are objects of  $\mathcal{C}_\varphi$  the tensor product  $V \otimes W$  and the dual  $V^*$  are still in  $\mathcal{C}_\varphi$ , namely one can define

$$a(v \otimes w) = \Delta_\varphi a(v \otimes w), \quad (af)v = f((Sa)v), \quad a \in U_q^\varphi(\mathfrak{g}), \quad v \in V, \quad w \in W, \quad f \in V^*.$$

Given  $V \in \mathcal{C}_\varphi$ , for a vector  $v \in V$  and a linear form  $f \in V^*$  we define the matrix coefficient  $c_{f,v}$  as follows :

$$c_{f,v} : U_q^\varphi(\mathfrak{g}) \longrightarrow K, \quad x \mapsto f(xv).$$

The  $K$ -module  $F_q^\varphi[G]$  spanned by all the matrix coefficients is equipped with the usual structure of dual Hopf algebra. The comultiplication  $\Delta$  (which doesn't depend on  $\varphi$ ) is given by :

$$(\Delta c_{f,v})(x \otimes y) = c_{f,v}(xy),$$

while the multiplication  $m_\varphi$  is given by :

$$m_\varphi(c_{f,v} \otimes c_{g,w}) = c_{f \otimes g, v \otimes w},$$

where  $V, W \in \mathcal{C}_\varphi$ ,  $v \in V, w \in W$ ,  $f \in V^*, g \in W^*$ ,  $x, y \in U_q^\varphi(\mathfrak{g})$ .

Moreover, since the algebras  $U_q^\varphi(\mathfrak{g})$  and  $U_q(\mathfrak{g})$  are equal, in order to obtain  $F_q^\varphi[G]$  (that as coalgebra is equal to  $F_q^0[G]$ ) it is enough to consider the subcategory of  $\mathcal{C}_\varphi$  given by the highest weight simple modules  $L(\Lambda)$ ,  $\Lambda \in P_+$ .

We recall that for these modules we have :

$$L(\Lambda) = \bigoplus_{\lambda \in \Omega(\Lambda) \subseteq P} L(\Lambda)_\lambda, \quad L(\Lambda)^* \simeq L(-\omega_0 \Lambda), \quad L(\Lambda)^*_{-\mu} = (L(\Lambda)_\mu)^*$$

and that

$$F_q[G] = \bigoplus_{\Lambda \in P_+} L(\Lambda) \otimes L(\Lambda)^*.$$

**2.2.** We want now to link the comultiplication  $\Delta_\varphi$  in  $U_q^\varphi(\mathfrak{g})$  and the multiplication  $m_\varphi$  in  $F_q^\varphi[G]$  with a bivector  $u \in \Lambda^2(\mathfrak{h})$ ,  $\mathfrak{h}$  being the Cartan subalgebra of  $\mathfrak{g}$  and to do this we firstly give the Drinfel'd definition of quantized universal enveloping algebra  $U_\hbar(\mathfrak{g})$ .

Let  $\mathbb{Q}[[\hbar]]$  be the ring of formal series in  $\hbar$ , then  $U_\hbar(\mathfrak{g})$  is the  $\mathbb{Q}[[\hbar]]$ -algebra generated, as an algebra complete in the  $\hbar$ -adic topology, by the elements  $E_i, F_i, H_i$  ( $i = 1, \dots, n$ ) and defining relations :

$$[H_i, H_j] = 0, [H_i, E_j] = a_{ij}E_j, [H_i, F_j] = -a_{ij}F_j$$

added to relations that we can deduce from (1.5) by replacing  $q$  with  $\exp(\frac{\hbar}{2})$  and  $K_i$  with  $\exp(\frac{\hbar}{2}d_iH_i)$ . ■

Put now

$$u = \sum_{i,j=1}^n d_j \frac{\hbar}{2} u_{ji} H_i \otimes H_j \in \Lambda^2(\mathfrak{h}),$$

where the matrix  $TU = (d_i u_{ij})$  is antisymmetric.

Then for all  $x \in U_q^\varphi(\mathfrak{g})$  using the identity [R]

$$\exp(-u)(\Delta_0 x) \exp(u) = \Delta_\varphi x$$

we can compute the  $\psi_{ij}$ 's, namely

$$U = A^{-1}XA^{-1}.$$

Moreover we get the following useful equality (see [L-S2]):

$$(2.1) \quad m_\varphi(c_{f_1, v_1} \otimes c_{f_2, v_2}) = q^{\frac{1}{2}((\varphi\mu_1, \mu_2) - (\varphi\lambda_1, \lambda_2))} m_0(c_{f_1, v_1} \otimes c_{f_2, v_2}),$$

for  $\Lambda_i \in P_+$ ,  $v_i \in L(\Lambda_i)_{\mu_i}$ ,  $f_i \in L(\Lambda_i)^*_{-\lambda_i}$ ,  $i = 1, 2$ .

Observe that (2.1) justifies the condition  $\frac{1}{2}(\varphi\lambda, \mu) \in \mathbb{Z}$ ,  $\forall \lambda, \mu \in P$ , required for  $\varphi$  (see (1.1)).

**2.3.** Since we are interested in the study at roots of 1 we need an integer form  $R_q^\varphi[G]$  of the multiparameter quantum function algebra. For this purpose define  $\Gamma^\varphi(\mathfrak{g})$  to be the  $R$ -subHopf algebra of  $U_q^\varphi(\mathfrak{g})$  generated by  $\Gamma^\varphi(\mathfrak{b}_+)$  and  $\Gamma^\varphi(\mathfrak{b}_-)$  and consider the subcategory  $\mathcal{D}_\varphi$  of  $\Gamma^\varphi(\mathfrak{g})$ -mod given by the free  $R$ -modules of finite rank in which  $K_i, \begin{pmatrix} K_i & 0 \\ t & \end{pmatrix}$  act by diagonal matrices with eigenvalues  $q_i^m, \begin{pmatrix} m \\ t \end{pmatrix}_{q_i}$ . Define  $R_q^\varphi[G]$  as the submodule generated by the matrix coefficients constructed with the objects of  $\mathcal{D}_\varphi$ . Similarly define  $R_q^\varphi[B_+]$  and  $R_q^\varphi[B_-]$  starting with opportune subcategories of  $\Gamma^\varphi(\mathfrak{b}_+) - mod$  and  $\Gamma^\varphi(\mathfrak{b}_-) - mod$  respectively.

In completely analogy with the case  $\varphi = 0$  and essentially in the same way (cf. prop.4.2 in [D-L]) we can prove that the pairings  $\pi'_\varphi, \overline{\pi}'_\varphi, \pi''_\varphi, \overline{\pi}''_\varphi$  induce the Hopf algebra isomorphisms

$$(2.2) \quad R_q^\varphi[B_\pm]' \simeq R_q^\varphi[B_\pm] \simeq R_q^\varphi[B_\pm]''$$

and in fact these isomorphisms are the motivations for having introduced the pairings.

**2.4.** Consider now the maps

$$\Gamma^\varphi(\mathfrak{b}_-) \otimes_R \Gamma^\varphi(\mathfrak{b}_+) \xrightarrow{\iota_-} \Gamma^\varphi(\mathfrak{g}) \otimes_R \Gamma^\varphi(\mathfrak{g}) \xrightarrow{m} \Gamma^\varphi(\mathfrak{g})$$

$$\Gamma^\varphi(\mathfrak{b}_+) \otimes_R \Gamma^\varphi(\mathfrak{b}_-) \xrightarrow{\iota_+} \Gamma^\varphi(\mathfrak{g}) \otimes_R \Gamma^\varphi(\mathfrak{g}) \xrightarrow{m} \Gamma^\varphi(\mathfrak{g})$$

where  $\iota_\pm$  are the natural embedding and  $m$  is the multiplication map. The corresponding dual maps composed with the isomorphisms (2.2) give the injections :

$$\mu'_\varphi : R_q^\varphi[G] \xrightarrow{\Delta_\varphi} R_q^\varphi[G] \otimes_R R_q^\varphi[G] \xrightarrow{r_-} R_q^\varphi[B_-] \otimes_R R_q^\varphi[B_+] \simeq R_q^\varphi[B_-]' \otimes_R R_q^\varphi[B_+]'$$

$$\mu''_\varphi : R_q^\varphi[G] \xrightarrow{\Delta_\varphi} R_q^\varphi[G] \otimes_R R_q^\varphi[G] \xrightarrow{r_+} R_q^\varphi[B_+] \otimes_R R_q^\varphi[B_-] \simeq R_q^\varphi[B_+]'' \otimes_R R_q^\varphi[B_-]''.$$

Let put, for  $M$  in  $\Gamma(\mathfrak{t})$ ,  $\lambda$  in  $P$ ,

$$M(\lambda) = \pi'_\varphi(M, K_{(1-\varphi)\lambda}) = \overline{\pi}'_\varphi(M, K_{-(1+\varphi)\lambda}) = \pi''_\varphi(K_{(1+\varphi)\lambda}, M) = \overline{\pi}''_\varphi(K_{-(1-\varphi)\lambda}, M) = \pi_0(M, K_\lambda).$$

It is now easy to prove the following (see Lemma 4.3.in [D-L])

**2.5. Lemma**

(i) *The image of  $\mu'_\varphi$  is contained in the  $R$ -subalgebra  $A'_\varphi$  generated by the elements*

$$e_\alpha^\varphi \otimes 1, 1 \otimes f_\alpha^\varphi, K_{(1-\varphi)\lambda} \otimes K_{-(1+\varphi)\lambda}, \lambda \in P, \alpha \in \Phi_+.$$

(ii) *The image of  $\mu''_\varphi$  is contained in the  $R$ -subalgebra  $A''_\varphi$  generated by the elements*

$$1 \otimes e_\alpha^\varphi, f_\alpha^\varphi \otimes 1, K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}, \lambda \in P, \alpha \in \Phi_+.$$

**2.6.** Define as in [D-L] the matrix coefficients  $\psi_{\pm\lambda}^{\pm\alpha}$ , that is for each  $\lambda \in P_+$  call  $v_\lambda$  (resp.  $v_{-\lambda}$ ) a choosen highest (resp. lowest ) weight vector of  $L(\lambda)$  (resp. of  $L(-\lambda)$ , the irreducible module of lowest weight  $-\lambda$ ). Let  $\phi_{\pm\lambda}$  the unique linear form on  $L(\pm\lambda)$  such that  $\phi_{\pm\lambda}v_{\pm\lambda} = 1$  and  $\phi_{\pm\lambda}$  vanishes on the unique  $\Gamma(\mathfrak{t})$ -invariant complement of  $Kv_{\pm\lambda} \subset L(\pm\lambda)$ .

For  $\rho = \sum_{i=1}^n \omega_i$ , put  $\psi_{\pm\rho} = c_{\phi_{\pm\rho}, v_{\pm\rho}}$ , and for  $\alpha \in \Phi_+$  define

$$\psi_\lambda^\alpha(x) = \phi_\lambda(E_\alpha x v_\lambda), \psi_\lambda^{-\alpha}(x) = \phi_\lambda(x F_\alpha v_\lambda),$$

$$\psi_{-\lambda}^{\alpha}(x) = \phi_{-\lambda}(xE_{\alpha}v_{-\lambda}), \quad \psi_{-\lambda}^{-\alpha}(x) = \phi_{-\lambda}(F_{\alpha}xv_{-\lambda}).$$

**2.7. Proposition** *The maps  $\mu'_{\varphi}$ ,  $\mu''_{\varphi}$  induce algebra isomorphisms*

$$R_q^{\varphi}[G][\psi_{\rho}^{-1}] \simeq A'_{\varphi}, \quad R_q^{\varphi}[G][\psi_{-\rho}^{-1}] \simeq A''_{\varphi}.$$

*Proof.* First of all we specify that what we want to prove is that the subalgebra generated by  $Im(\mu'_{\varphi})$  and  $\mu'_{\varphi}(\psi_{\rho}^{-1})$  is indeed  $A'_{\varphi}$  and similarly for  $A''_{\varphi}$ . Consider the case of  $\mu'_{\varphi}$ . First of all we have

$$\mu'_{\varphi}(\psi_{\rho}) = K_{(1-\varphi)\rho} \otimes K_{-(1+\varphi)\rho}.$$

Moreover an easy calculation gives

$$\mu'_{\varphi}(\psi_{\omega_i}^{\alpha_i}) = -q^{-\frac{1}{2}(\varphi\alpha_i, \omega_i)} e_i^{\varphi} K_{(1-\varphi)\omega_i} \otimes K_{-(1+\varphi)\omega_i},$$

from which we get  $e_i^{\varphi} \otimes 1 \in Im(\mu'_{\varphi}, \mu'_{\varphi}(\psi_{\rho}^{-1}))$ .

To see that  $e_{\alpha}^{\varphi} \otimes 1 \in Im(\mu'_{\varphi}, \mu'_{\varphi}(\psi_{\rho}^{-1}))$  we procede as in [D-L], by induction on  $ht(\alpha)$ , namely

$$\mu'_{\varphi}(\psi_{\lambda}^{\alpha}) = (-q^{-(\tau_{\alpha}, \lambda)} x(\alpha, \lambda) e_{\alpha}^{\varphi} + d) K_{(1-\varphi)\lambda} \otimes K_{-(1+\varphi)\lambda}$$

where  $d$  is a  $R$ -linear combination of monomials of degree  $\alpha$  in  $e_{\beta}^{\varphi}$  with  $ht(\beta) < ht(\alpha)$  and

$$x(\alpha, \lambda) = \frac{q^{(\alpha, \lambda)} - q^{-(\alpha, \lambda)}}{q_{\alpha} - q_{\alpha}^{-1}}.$$

Similar arguments hold for  $1 \otimes f_{\alpha}^{\varphi}$ . □

### 3. Roots of one

**3.1.** Consider a primitive  $l$ -th root of unity  $\varepsilon$  with  $l$  a positive odd integer prime to 3 if  $\mathfrak{g}$  is of type  $G_2$  and define  $\Gamma_{\varepsilon}^{\varphi}(\mathfrak{g}) = \Gamma^{\varphi}(\mathfrak{g}) \otimes_R \mathbb{Q}(\varepsilon)$ ,  $F_{\varepsilon}^{\varphi}[G] = R_q^{\varphi}[G] \otimes_R \mathbb{Q}(\varepsilon)$ ,  $\psi : R_q^{\varphi}[G] \rightarrow F_{\varepsilon}^{\varphi}[G]$ ,  $\psi(c_{f,v}) = \bar{c}_{f,v}$ , the canonical projection. By abuse of notations, the image in  $\Gamma_{\varepsilon}^{\varphi}(\mathfrak{g})$  of an element of  $\Gamma^{\varphi}(\mathfrak{g})$  will be indicated with the same symbol.

Remark that for  $q = l = 1$  the quotient of  $\Gamma_1^{\varphi}(\mathfrak{g})$  by the ideal generated by the  $(K_i - 1)$ 's is isomorphic, as Hopf algebra, to the usual enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g}$  over the field  $\mathbb{Q}$ ; while the Hopf algebra  $F_1^{\varphi}[G]$  is isomorphic to the coordinate ring  $\mathbb{Q}[G]$  of  $G$ .

**3.2.** It is important to stress some results of Lusztig [L1] and De Concini-Lyubashenko [D-L] in the case  $\varphi = 0$  which still hold in our case principally by virtue of formulas (1.9). More precisely :

(i) There exists an epimorphism of Hopf algebras (use (1.9))  $\phi : \Gamma^{\varphi}(\mathfrak{g}) \rightarrow U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)}$  relative to  $R \rightarrow \mathbb{Q}(\varepsilon)$  such that  $(i = 1, \dots, n; p > 0) :$

$$\phi E_i^{(p)} = e_i^{(\frac{p}{l})}, \quad \phi F_i^{(p)} = f_i^{(\frac{p}{l})}, \quad \phi \left( \begin{matrix} K_i; 0 \\ p \end{matrix} \right) = \left( \begin{matrix} h_i \\ \frac{p}{l} \end{matrix} \right) \text{ (if } l|p, 0 \text{ otherwise); } \phi K_i = 1, \quad \phi q = \varepsilon.$$

Here  $e_i, f_i, h_i$  are Chevalley generators for  $\mathfrak{g}$ . Generators for the kernel  $J$  of  $\phi$  are the elements :

$$E_i^{(p)}, F_i^{(p)}, \binom{K_i; 0}{p}, K_i - 1, p_l(q) \ (i = 1, \dots, n; p > 0; l \nmid p).$$

Moreover if  $\Gamma_l$  is the free  $R$ -module with basis

$$\prod_{\beta} F_{\beta}^{(m_{\beta})} \xi_t \prod_{\alpha} E_{\alpha}^{(n_{\alpha})}, \quad m_{\beta}, t_i, n_{\alpha} \equiv 0 \pmod{l},$$

then  $U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)} \simeq \Gamma_l / p_l(q) \Gamma_l$ .

(ii) Denote by  $I$  the ideal of  $\Gamma_{\varepsilon}^{\varphi}(\mathfrak{g})$  generated by  $E_i, F_i, K_i - 1$  ( $i = 1, \dots, n$ ). The elements  $\prod_{\beta} F_{\beta}^{(n_{\beta})} M \prod_{\alpha} E_{\alpha}^{(m_{\alpha})}$ , where  $M$  is in the ideal  $(K_i - 1 | i = 1, \dots, n) \subset \Gamma_{\varepsilon}(\mathfrak{t})$  or one of the exponents  $n_{\beta}, m_{\alpha}$  is not divisible by  $l$ , constitute an  $R$ -basis of  $I$ . The epimorphism  $\phi$  induces the Hopf algebras isomorphism  $U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)} \simeq \Gamma_{\varepsilon}^{\varphi}(\mathfrak{g})/I$  and an  $R$ -basis for  $U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)}$  is given by the elements

$$\prod_{\beta} F_{\beta}^{(n_{\beta})} M \prod_{\alpha} E_{\alpha}^{(m_{\alpha})}, \quad n_{\beta}, m_{\alpha} \equiv 0 \pmod{l}, \quad M \text{ polynomial in } \binom{K_i; 0}{l}.$$

**3.3.** An important consequence of 3.2. is the existence of a central Hopf subalgebra  $F_0$  of  $F_{\varepsilon}^{\varphi}[G]$  which is naturally isomorphic to  $\mathbb{Q}(\varepsilon)[G]$ . An element of  $F_{\varepsilon}^{\varphi}[G]$  belongs to  $F_0$  if and only if it vanishes on  $I$  and we deduce from [L1] that

$$(3.1) \quad F_0 = \langle \bar{c}_{f,v} \mid f \in L(l\Lambda)_{-l\nu}^*, v \in L(l\Lambda)_{l\mu}; \nu, \mu \in P_+ \rangle,$$

where  $\langle \rangle$  denotes the  $\mathbb{Q}(\varepsilon)$ -span.

**3.4. Lemma** *Let  $\bar{c}_{f,v}$  be an element of  $F_0$  and  $\bar{c}_{g,w}$  an element of  $F_{\varepsilon}^{\varphi}[G]$ . Then*

$$m_{\varphi}(\bar{c}_{f,v} \otimes \bar{c}_{g,w}) = m_0(\bar{c}_{f,v} \otimes \bar{c}_{g,w}).$$

*Proof.* It is enough to consider identity (3.1) and to apply formula (2.1). □

**3.5. Proposition**  *$F_{\varepsilon}^{\varphi}[G]$  is a projective module over  $F_0$  of rank  $l^{\dim G}$ .*

*Proof.* By 3.4.  $F_{\varepsilon}^{\varphi}[G]$  and  $F_{\varepsilon}^0[G]$  are the same  $F_0$ -modules and so the result follows from [D-L]. □

**3.6.** Define  $A_{\varepsilon}^{\varphi} = A_{\varphi}'' \otimes_R \mathbb{Q}(\varepsilon)$ . Let  $\mu_{\varepsilon}^{\varphi} : F_{\varepsilon}^{\varphi}[G] \rightarrow A_{\varepsilon}^{\varphi}$  be the injection induced by  $\mu_{\varphi}''$ ; we get the isomorphism (see 2.7.)  $F_{\varepsilon}^{\varphi}[G][\psi_{-l\rho}^{-1}] \simeq A_{\varepsilon}^{\varphi}$ . Denote by  $A_0^{\varphi}$  the subalgebra of  $A_{\varepsilon}^{\varphi}$  generated by

$$1 \otimes (e_{\alpha}^{\varphi})^l, (f_{\alpha}^{\varphi})^l \otimes 1, K_{-(1+\varphi)(l\lambda)} \otimes K_{(1-\varphi)(l\lambda)} \ (\alpha \in \Phi_+, \lambda \in P),$$

then  $\mu_\varepsilon^\varphi(F_0)[\psi_{-l\rho}^{-1}] = A_0^\varphi$  (it is a consequence of 3.2.,3.3.).

**3.7.** A basis for  $A_\varepsilon^\varphi$  is the following

$$(F_{\beta_N} K_{\tau_{\beta_N}})^{n_N} \cdots (F_{\beta_1} K_{\tau_{\beta_1}})^{n_1} K_{-(1+\varphi)\omega_1}^{s_1} \cdots K_{-(1+\varphi)\omega_n}^{s_n} \otimes K_{(1-\varphi)\omega_1}^{s_1} \cdots K_{(1-\varphi)\omega_n}^{s_n} (E_{\beta_1} K_{\tau_{\beta_1}})^{m_1} \cdots (E_{\beta_N} K_{\tau_{\beta_N}})^{m_N}.$$

Moreover  $A_\varepsilon^\varphi$  is a maximal order in its quotient division algebra. We can prove this following the ideas in [D-P1], th. 6.5. (cf also [D-K-P1]).

**3.8. Theorem**  $F_\varepsilon^\varphi[G]$  is a maximal order in its quotient division algebra.

*Proof.* In order to repeat the reasoning in th.7.4. of [D-L] we need elements  $x_1, \dots, x_r$  in  $F_0$  such that  $(x_1, \dots, x_r) = (1)$  and  $F_\varepsilon^\varphi[G][x_i^{-1}]$  is finite over  $F_0[x_i^{-1}]$ . In fact, when  $\varphi \neq 0$  we cannot use left translations (by elements of  $W$ ) of  $\psi_{-l\rho}$ . For  $g \in G$ , let  $\mathcal{M}_g$  be the maximal ideal in  $\mathbb{Q}(\varepsilon)[G]$  determined by it. Then  $(F_\varepsilon^\varphi[G])_{\mathcal{M}_g}$  is a free  $(F_0)_{\mathcal{M}_g}$ -module of finite type (by 3.5.) and there exists  $x_g \in F_0 \setminus \mathcal{M}_g$  (that is  $x_g(g) \neq 0$ ) such that  $F_\varepsilon^\varphi[G][x_g^{-1}]$  is a free  $F_0[x_g^{-1}]$ -module of finite type. Now  $G = \bigcup_g D(x_g)$ , where  $D(x_g) = \{x \in G \mid x_g(x) \neq 0\}$ , and so there exist  $x_1, \dots, x_r \in F_0$  for which  $G = \bigcup_{i=1}^r D(x_i)$ , that is the assert.  $\square$

#### 4. Poisson structure of G

**4.1.** To the quantization  $\Gamma^\varphi(\mathfrak{g})$  of  $U(\mathfrak{g})_{\mathbb{Q}(\varepsilon)}$  is associated, in the sense of [D2], a Manin triple  $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}_\varphi)$  and a Poisson Hopf algebra structure on  $F_0 = \mathbb{Q}(\varepsilon)[G]$ .

The Manin triple is composed of  $\mathfrak{g}$ , identified with the diagonal subalgebra of  $\mathfrak{d} = \mathfrak{g} \times \mathfrak{g}$ , and of  $\mathfrak{g}_\varphi = \mathfrak{c}_\varphi \oplus \mathfrak{u}$ , where  $\mathfrak{c}_\varphi = \{(-x + \varphi(x), x + \varphi(x)) \mid x \in \mathfrak{h}\}$ ,  $\mathfrak{u} = (\mathfrak{n}_+ \times \mathfrak{n}_-)$ ,  $\mathfrak{n}_\pm$  is the nilpotent radical of a fixed Borel subalgebra  $\mathfrak{b}_\pm$  of  $\mathfrak{g}$ . Here we denote, by abuse of notation, again by  $\varphi$  the endomorphism of  $\mathfrak{h}$  obtained by means of the identification  $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$  with the Killing form. The bilinear form on  $\mathfrak{d}$ , for which  $\mathfrak{g}$  and  $\mathfrak{g}_\varphi$  become isotropic Lie subalgebras, is defined by

$$\prec(x, y), (x', y') \succ = \langle x, x' \rangle - \langle y, y' \rangle,$$

where  $\langle, \rangle$  is the Killing form on  $\mathfrak{g}$ .

In order to define a bracket  $\{, \}_\varphi$  on  $F_0$  we can proceed as in [D-L], namely lemma 8.1. still hold after substitution  $\Delta \leftrightarrow \Delta_\varphi$ . We want here to give also a direct construction starting from the bracket  $\{, \}_0$  corresponding to  $\varphi = 0$ .

**4.2. Proposition** Let  $\Lambda_i \in P_+$ ,  $v_i \in L(l\Lambda_i)_{l\mu_i}$ ,  $f_i \in L(l\Lambda_i)^*_{-l\lambda_i}$ ,  $c_i = c_{f_i, v_i}$ ,  $i = 1, 2$  and define  $\chi(1, 2) = \frac{1}{2}((\varphi\mu_1, \mu_2) - (\varphi\lambda_1, \lambda_2)) = -\chi(2, 1)$ . Then :

$$\{\bar{c}_1, \bar{c}_2\}_\varphi = \{\bar{c}_1, \bar{c}_2\}_0 + 2\chi(1, 2)m_\varphi(\bar{c}_1 \otimes \bar{c}_2).$$

*Proof.* Let  $[\cdot, \cdot]_\varphi$  be the commutator in the algebra  $R_q^\varphi[G]$ . The using (2.1) we obtain :

$$[c_1, c_2]_\varphi - [c_1, c_2]_0 = (q^{l^2\chi(1,2)} - 1)m_0(c_1 \otimes c_2) - (q^{l^2\chi(2,1)} - 1)m_0(c_2 \otimes c_1).$$

Now we recall that, by construction (see [D-L]), if  $[c_1, c_2]_\varphi = p_l(q)c$ , we put

$$\{\bar{c}_1, \bar{c}_2\}_\varphi = \left(\frac{p_l(q)}{l(q^l - 1)}\right)_{|q=\varepsilon} \bar{c},$$

and that (by 3.4.) in  $F_0$ ,  $m_\varphi$  coincides with  $m_0$ . Then, by projecting in  $F_\varepsilon^\varphi[G]$  and using the commutativity in  $F_0$ , we get :

$$\{\bar{c}_1, \bar{c}_2\}_\varphi = \{\bar{c}_1, \bar{c}_2\}_0 + (h_{12} - h_{21})m_\varphi(\bar{c}_1 \otimes \bar{c}_2),$$

where

$$h_{ij} = \left(\frac{q^{l^2\chi(i,j)} - 1}{p_l(q)} \cdot \frac{p_l(q)}{l(q^l - 1)}\right)_{|q=\varepsilon} = \left(\frac{q^{l^2\chi(i,j)} - 1}{l(q^l - 1)}\right)_{|q=\varepsilon}.$$

Define

$$p(x) = \frac{x^{l\chi(1,2)} - x^{-l\chi(1,2)}}{l(x - 1)} = \frac{x^{-l\chi(1,2)}}{l} \left( \sum_{k=0}^{2l\chi(1,2)-1} x^k \right) \in \mathbb{Q}[x, x^{-1}],$$

then  $h_{12} - h_{21} = p(1) = 2\chi(1, 2)$  and we are done.  $\square$

**4.3. Corollary** (i) Any function  $\{\bar{c}_1, \bar{c}_2\}_\varphi$ ,  $\bar{c}_i \in F_0$ , vanishes on the torus  $T = \exp \mathfrak{h} \subseteq G$ .  
(ii) Right and left shift by an element of the torus are automorphisms of the Poisson algebra  $\mathbb{Q}(\varepsilon)[G]$ .

*Proof.* (i) In [D-L] the assert is proved for  $\{, \}_0$  then, by 4.2., we only need to prove that  $2\chi(1, 2)m_\varphi(\bar{c}_1 \otimes \bar{c}_2)$  vanishes in the elements of torus. An easy calculation shows (here we use the identification  $h_i \leftrightarrow \begin{pmatrix} K_i; 0 \\ l \end{pmatrix}$  in agreement with 3.1.(i)) that, for  $t \in T$ ,  $(\bar{c}_1 \otimes \bar{c}_2)(\Delta_\varphi t) \neq 0$  if  $\lambda_i = \mu_i$ , that is if  $\chi(1, 2) = 0$ .

(ii) The right shift by the element  $t \in T$  is defined as the element  ${}^t\bar{c} = \bar{c}_{(1)} \cdot \bar{c}_{(2)}(t)$  (similarly for the left shift) and then the claim follows from (i) and from formal properties of the bracket in a Poisson Hopf algebra.  $\square$

**4.4.** Let  $T, C_\varphi, U_\pm, B_\pm$ , be the closed connected subgroups of  $G$  associated to  $\mathfrak{h}, \mathfrak{c}_\varphi, \mathfrak{n}_\pm, \mathfrak{b}_\pm$  and let  $D$  be  $G \times G$ . Put :  $G_\varphi = C_\varphi(U_+ \times U_-)$ ,  $H = \{(x, x) \mid x \in T\}$ ,  $\bar{H} = \{(x, x) \mid x \in \mathfrak{h}\}$ . We have the Bruhat decomposition

$$D = \bigcup_{w \in W \times W} HG_\varphi w G_\varphi.$$

The symplectic leaves, that is the maximal connected symplectic subvarieties of  $G$ , are the connected components, all isomorphic, of  $X_w^\varphi = p^{-1}(G_\varphi \backslash G_\varphi w H G_\varphi)$  for  $w$  running in  $W \times W$ , where  $p : G \hookrightarrow D \rightarrow G_\varphi \backslash D$  is the diagonal immersion followed by the canonical projection (see [L-W]). Moreover  $X_w^\varphi$  are the minimal  $T$ -biinvariant Poisson submanifolds of  $G$ . Observe that  $\mathfrak{c}_\varphi + \bar{\mathfrak{h}} = \mathfrak{c}_0 + \bar{\mathfrak{h}}$  and so  $C_\varphi H = C_0 H$ , that is  $X_w^\varphi = X_w^0 = X_w = (B_+ w_1 B_+) \cap (B_- w_2 B_-)$  for all  $w = (w_1, w_2) \in W \times W$ .

**4.5. Proposition** *Let  $w = (w_1, w_2) \in W \times W$ . The dimension of a symplectic leaf in  $X_{w_1, w_2}$  is equal to*

$$l(w_1) + l(w_2) + rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi)),$$

where  $l(\cdot)$  is the lenght function on  $W$ .

*Proof.* Since  $p$  is an unramified finite covering of its image, it is enough to calculate the dimension of the  $G_\varphi$ -orbits in  $G_\varphi \backslash D$ . Moreover  $G_\varphi \subseteq B_+ \times B_- = B$ , then we can consider the map  $\pi : G_\varphi \backslash D \rightarrow B \backslash D$ , equivariant for the right action of  $G_\varphi$  and so preserving  $G_\varphi$ -orbits. In  $B \backslash D$  the  $G_\varphi$ -orbits coincide with the  $B$ -orbits which are equals to  $B \backslash (B_+ w_1 B_+ \times B_- w_2 B_-) = \Theta(w_1, w_2)$ . Note that  $\pi$  is a principal  $T/\Gamma$ -bundle, where  $\Gamma = \{t \in T \mid t^2 = 1\}$ . Let  $\Theta$  be a  $G_\varphi$ -orbit in  $D$  such that  $\pi(\Theta) = \Theta(w_1, w_2)$ , then  $\pi|_\Theta : \Theta \rightarrow \Theta(w_1, w_2)$  is a principal  $T_{w_1, w_2}/\Gamma$ -bundle where  $T_{w_1, w_2} = \{t \in T \mid t\Theta = \Theta\}$ . From it follows  $\dim \Theta = \dim(T_{w_1, w_2}/\Gamma) + \dim \Theta(w_1, w_2)$ , that is

$$\dim \Theta = \dim T_{w_1, w_2} + l(w_1) + l(w_2).$$

In order to calculate  $\dim T_{w_1, w_2}$  take  $n_1, n_2$  representatives of  $w_1, w_2$  in the normalizer of  $T$ . We get  $t\Theta = \Theta$  if and only if there exist  $(t_1, t_2), (s_1, s_2) \in C_\varphi$  and  $(s_1, s_2)(t, t) = (n_1, n_2)(t_1, t_2)(n_1, n_2)^{-1}$ . Let  $u, v$  be elements in  $\mathfrak{h}$  such that

$$(s_1, s_2) = (\exp(-u + \varphi u), \exp(u + \varphi u)), \quad (t_1, t_2) = (\exp(-v + \varphi v), \exp(v + \varphi v)).$$

We are so reduced to find  $x \in \mathfrak{h}$  for which

$$\begin{cases} -u + \varphi u + x = w_1(-v + \varphi v) \\ u + \varphi u + x = w_2(v + \varphi v) \end{cases},$$

that is

$$\begin{cases} 2x + 2\varphi u = (-w_1(1 - \varphi) + w_2(1 + \varphi))v \\ 2u = (w_1(1 - \varphi) + w_2(1 + \varphi))v \end{cases}.$$

We find  $2x = ((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi))v$  and so

$$\dim T_{w_1, w_2} = rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi)).$$

□

## 5. Representations

**5.1.** In all this paragraph we shall substitute the basic field  $\mathbb{Q}(\varepsilon)$  with  $\mathbb{C}$ . The fact that  $F_\varepsilon^\varphi[G]$  is a projective module of rank  $l^{dim(G)}$  over  $F_0$  allows us to define a bundle of algebras on  $G$  with fibers  $F_\varepsilon^\varphi[G](g) = F_\varepsilon^\varphi[G]/\mathcal{M}_g F_\varepsilon^\varphi[G]$  (for more details on this construction confront section 9 in [D-L]). From the results of previous chapter also in our case the algebras  $F_\varepsilon^\varphi[G](g)$  and  $F_\varepsilon^\varphi[G](h)$  are isomorphic for  $g, h$  in the same  $X_{w_1, w_2}$  that is, using the central character map  $Spec(F_\varepsilon^\varphi[G]) \rightarrow Spec(F_0) = G$ , the representation theory of  $F_\varepsilon^\varphi[G]$  is constant on the sets  $X_{w_1, w_2}$ .

**5.2.** Let  $w_1, w_2$  be two elements in  $W$ . Choose reduced expressions for them, namely  $w_1 = s_{i_1} \cdots s_{i_t}$ ,  $w_2 = s_{j_1} \cdots s_{j_m}$ , and consider the corresponding ordered sets of positive roots  $\{\beta_1, \dots, \beta_t\}$  and  $\{\gamma_1, \dots, \gamma_m\}$  with  $\beta_1 = \alpha_{i_1}$ ,  $\beta_r = s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}$  for  $r > 1$  and similarly for the  $\gamma_i$ 's. Define  $A_{\varepsilon, \varphi}^{(w_1, w_2)}$  as the subalgebra in  $A_{\varepsilon}^{\varphi}$  generated by the elements

$$1 \otimes e_{\beta_i}, f_{\gamma_j} \otimes 1, K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda} \quad (i = 1, \dots, t, j = 1, \dots, m, \lambda \in P),$$

and put  $A_{0, \varphi}^{(w_1, w_2)} = A_{\varepsilon, \varphi}^{(w_1, w_2)} \cap A_0^{\varphi}$ . Note that these definitions do not depend on the reduced expressions (see [D-K-P2]). The algebra  $A_{\varepsilon, \varphi}^{(w_1, w_2)}$  is a free module of rank  $l^{(w_1)} + l^{(w_2)} + n$  over its central subalgebra  $A_{0, \varphi}^{(w_1, w_2)}$  and so it is finite over its centre and has finite degree. We will call  $d_{\varphi}(w_1, w_2)$  the degree of  $A_{\varepsilon, \varphi}^{(w_1, w_2)}$ .

**5.3.** There is an algebra isomorphism  $A_{0,0}^{(w_1, w_2)} \simeq A_{0, \varphi}^{(w_1, w_2)}$  induced by the isomorphism between the algebras  $A_0^0$  and  $A_0^{\varphi}$  given by

$$1 \otimes e_{\alpha}^l \mapsto 1 \otimes (e_{\alpha}^{\varphi})^l, f_{\alpha}^l \otimes 1 \mapsto (f_{\alpha}^{\varphi})^l \otimes 1, K_{-l\lambda} \otimes K_{l\lambda} \mapsto K_{-(1+\varphi)l\lambda} \otimes K_{(1-\varphi)l\lambda}.$$

Therefore  $\text{Spec}(A_{0, \varphi}^{(w_1, w_2)})$  is birationally isomorphic to  $X_{w_1, w_2} \cap \text{Spec}(A_0^{\varphi})$  (cf prop. 10.4 in [D-L]). From this, and reasoning as in [D-L], it follows that the dimension of any representation of  $F_{\varepsilon}^{\varphi}[G]$  lying over a point in  $X_{w_1, w_2}$  has dimension divisible by  $d_{\varphi}(w_1, w_2)$ .

**5.4.** In order to calculate the degree  $d_{\varphi}(w_1, w_2)$  we introduce another set of generators for  $A_{\varepsilon, \varphi}^{(w_1, w_2)}$ . Call  $\Xi$  the antisomorphism of algebras  $\Xi : U_q^{\varphi}(\mathfrak{g}) \rightarrow U_q^{\varphi}(\mathfrak{g})$  which is the identity on  $E_i, F_i, q$  and send  $K_i$  into  $K_i^{-1}$ ; we get  $T_i^{-1} = \Xi T_i \Xi$  (see [L2]). For  $\alpha \in \{\gamma_1, \dots, \gamma_m\}$ , let  $\Xi(f_{\alpha}^0)K_{\tau_{\alpha}} = f_{\alpha}^{\varphi'}$ . Observe that, for  $r = 1, \dots, m$ ,  $\Xi(F_{\gamma_r}) = T_{i_1}^{-1} \cdots T_{i_{r-1}}^{-1}(F_{i_r})$ . We want now to show that the sets  $\{f_{\beta_i}^{\varphi}, | i = 1, \dots, m\}$  and  $\{f_{\beta_i}^{\varphi'}, | i = 1, \dots, m\}$  generate the same subalgebra of  $R_q^{\varphi}[B_+]''$ . Let  $H_1, H_2$  be the subalgebras respectively generated by these sets. From  $\Xi(U_q^{\varphi}(\mathfrak{n}_-)) = U_q^{\varphi}(\mathfrak{n}_-)$  and  $T_i^{\pm 1}(R_q^{\varphi}[B_+]') \subseteq R_q^{\varphi}[B_+]''$  for every  $i$ , follow that  $f_{\alpha}^{\varphi'}$  belongs to the algebra generated by  $\{f_{\gamma_i}^0 | i = 1, \dots, m\}$  and

$$f_{\alpha}^{\varphi'} = f_{\alpha}^0 K_{\tau_{\alpha}} = \left( \sum_s c_s (f_{\gamma_m}^0)^{s_m} \cdots (f_{\gamma_1}^0)^{s_1} \right) K_{\tau_{\alpha}} = \sum_s c'_s (f_{\gamma_m}^{\varphi})^{s_m} \cdots (f_{\gamma_1}^{\varphi})^{s_1} \in H_1,$$

where  $c'_s = q^{r_s} c_s$  for an integer  $r_s$ . In a similar way we can show that  $f_{\alpha}^{\varphi} \in H_2$ .

Put now, in  $A_{\varepsilon, \varphi}^{(w_1, w_2)}$ ,

$$x'_i = 1 \otimes e_{\beta_i}^{\varphi} \quad (i = 1, \dots, t), \quad y'_j = f_{\gamma_j}^{\varphi'} \otimes 1 \quad (j = 1, \dots, m), \quad z'_r = K_{-(1+\varphi)\omega_r} \otimes K_{(1-\varphi)\omega_r} \quad (r = 1, \dots, n).$$

As in the case  $\varphi = 0$ ,  $A_{\varepsilon, \varphi}^{(w_1, w_2)}$  is an iterated twisted polynomial algebra and the corresponding quasipolynomial algebra is generated by elements  $x_i, y_j, z_r$  with relations which are easily found (see [L-S1] and [D-K-P2]). Namely :

$$x_i x_j = \varepsilon^{(\beta_i, (1+\varphi)\beta_j)} x_j x_i \quad (1 \leq j < i \leq t), \quad y_i y_j = \varepsilon^{-(\gamma_i, (1+\varphi)\gamma_j)} y_j y_i \quad (1 \leq j < i \leq m),$$

$$z_i z_j = z_j z_i \ (1 \leq i, j \leq n), \ x_i y_j = y_j x_i \ (1 \leq i \leq t, i \leq j \leq m),$$

$$z_i x_j = \varepsilon^{((1-\varphi)\omega_i, \beta_j)} x_j z_i \ (1 \leq i \leq n, i \leq j \leq t), \ z_i y_j = \varepsilon^{((1+\varphi)\omega_i, \gamma_j)} y_j z_i \ (1 \leq i \leq n, i \leq j \leq m).$$

**5.5.** Let  $\mathbb{Z}_\varphi$  be the ring  $\mathbb{Z}[(2d_1 \cdots d_n \det(1-\varphi))^{-1}]$  and denote by  $\vartheta$  the isometry  $(1+\varphi)(1-\varphi)^{-1}$ . For each pair  $(w_1, w_2)$  in  $W \times W$ , consider the map  $e_\varphi(w_1, w_2) = 1 - w_1^{-1} \vartheta^{-1} w_2 \vartheta : P \otimes_{\mathbb{Z}} \mathbb{Q} \leftrightarrow Q \otimes_{\mathbb{Z}} \mathbb{Q}$ . Define  $l(\varphi)$  to be the least positive integer for which, for every  $(w_1, w_2)$ , the image of  $P \otimes_{\mathbb{Z}_\varphi} [l(\varphi)^{-1}]$  is a split summand of  $Q \otimes_{\mathbb{Z}_\varphi} [l(\varphi)^{-1}]$  (in special cases, namely when  $\vartheta$  fix the set of roots, one can explicitly take  $l(\varphi) = a_1 \cdots a_n$ , where  $\sum_{i=1}^n a_i \alpha_i$  is the longest root, as in [D-K-P2], but in general we need a case by case analysis).

An integer  $l$  is said to be a  $\varphi$ -good integer if, besides being prime to the  $2d_i$ , it is prime to  $\det(1-\varphi)$  and  $l(\varphi)$ .

**5.6. Theorem** *Let  $l$  be a  $\varphi$ -good integer,  $l > 1$ . Then,*

$$d_\varphi(w_1, w_2) = l^{\frac{1}{2}(l(w_1) + l(w_2) + rk((1-\varphi)w_1(1+\varphi) - (1+\varphi)w_2(1-\varphi)))}.$$

*Proof.* We work over  $S = \mathbb{Z}_\varphi[l(\varphi)^{-1}]$ . Let  $w_1, w_2$  be in  $W$ . Consider free  $S$ -modules  $V_{w_1}, V_{w_2}$  with basis  $u_1, \dots, u_t$  and  $v_1, \dots, v_m$  respectively. Define on  $V_{w_1}, V_{w_2}$  skew symmetric bilinear forms by

$$\langle u_i | u_j \rangle = (\beta_i, (1+\varphi)\beta_j) \ (1 \leq j < i \leq t), \ \langle v_i | v_j \rangle = (\gamma_i, (1+\varphi)\gamma_j) \ (1 \leq j < i \leq m),$$

and denote by  $C_{w_1}^\varphi, C_{w_2}^\varphi$  their matrices in the bases of the  $u_i$ 's and  $v_j$ 's respectively. Finally let  $D_{w_1}^\varphi, D_{w_2}^\varphi$  be the  $t \times n, m \times n$  matrices whose entries are  $(\beta_i, (1-\varphi)\omega_j)$  and  $(\gamma_i, (1+\varphi)\omega_j)$  respectively. Put

$$\Delta_{w_1, w_2}^\varphi = \begin{pmatrix} C_{w_1}^\varphi & 0 & D_{w_1}^\varphi \\ 0 & -C_{w_2}^\varphi & D_{w_2}^\varphi \\ -{}^t D_{w_1}^\varphi & -{}^t D_{w_2}^\varphi & 0 \end{pmatrix}, \ \Delta = \begin{pmatrix} C_{w_1}^0 & 0 & D_{w_1}^\varphi \\ 0 & -C_{w_2}^0 & D_{w_2}^\varphi \\ -{}^t D_{w_1}^\varphi & -{}^t D_{w_2}^\varphi & 0 \end{pmatrix}.$$

We want first of all prove that  $\Delta_{w_1, w_2}^\varphi$  is equivalent to  $\Delta$ , that is we want to exhibit an  $n \times t$  matrix  $M_1$  and an  $m \times n$  matrix  $M_2$  for which :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & M_2 \\ 0 & 0 & 1 \end{pmatrix} \Delta_{w_1, w_2}^\varphi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ M_1 & 0 & 1 \end{pmatrix} = \Delta,$$

or equivalently for which

$$C_{w_1}^\varphi + D_{w_1}^\varphi M_1 = C_{w_1}^0, \ C_{w_2}^\varphi + M_2 ({}^t D_{w_2}^\varphi) = C_{w_2}^0, \ D_{w_2}^\varphi M_1 = M_2 ({}^t D_{w_1}^\varphi).$$

First of all we need some notations. If  $f : V_1 \rightarrow V_2$  is a linear map and  $\mathcal{B}_1$  (resp.  $\mathcal{B}_2$ ) is a basis of  $V_1$  (resp.  $V_2$ ) we will indicate by  $M(f, \mathcal{B}_1, \mathcal{B}_2)$  the matrix of  $f$  in these given bases. Let now

$\check{\alpha}_i = \frac{\alpha_i}{d_i}$  and denote by  $\nu : \text{SP} \longrightarrow (\text{SP})^*$  the map given by  $\nu(\check{\alpha}_i) = \omega_i^*$  or, equivalently, the map which send  $\alpha_i$  to the linear form  $(\alpha_i, \cdot)$ . We define the following maps :

$$\begin{aligned} c_1^\varphi : V_{w_1} &\longrightarrow V_{w_1}^*, \quad u_j \mapsto \sum_{i>j} (\beta_j, (1+\varphi)\beta_i)u_i^* - \sum_{i<j} (\beta_i, (1+\varphi)\beta_j)u_i^*; \\ c_2^\varphi : V_{w_2} &\longrightarrow V_{w_2}^*, \quad v_j \mapsto \sum_{i>j} (\gamma_j, (1+\varphi)\gamma_i)v_i^* - \sum_{i<j} (\gamma_i, (1+\varphi)\gamma_j)v_i^*; \\ d_1^\varphi : \mathbb{Z}'\text{P} &\longrightarrow V_{w_1}^*, \quad \omega_i \mapsto \sum_j ((1+\varphi)\beta_j, \omega_i)u_j^*; \quad d_2^\varphi : \mathbb{Z}'\text{P} \longrightarrow V_{w_2}^*, \quad \omega_i \mapsto \sum_j ((1-\varphi)\gamma_j, \omega_i)v_j^*; \\ h_1 : V_{w_1} &\longrightarrow \text{SP}, \quad u_j \mapsto \beta_j; \quad h_2 : V_{w_2} \longrightarrow \text{SP}, \quad v_j \mapsto \gamma_j. \end{aligned}$$

Then we have

$$d_1^0 = \nu h_1, \quad d_2^0 = \nu h_2, \quad d_1^\varphi = d_1^0(1-\varphi), \quad d_2^\varphi = d_2^0(1+\varphi),$$

and we can easy verify that

$$c_1^\varphi - c_1^0 = -d_1^0\varphi h_1, \quad c_2^\varphi - c_2^0 = -d_2^0\varphi h_2.$$

Moreover, if  $Z = M(\varphi, \{\omega_i\}, \{\omega_i\})$ , we get

$$\begin{aligned} C_{w_1}^\varphi &= M(c_1^\varphi, \{u_i\}, \{u_i^*\}), \quad C_{w_2}^\varphi = M(c_2^\varphi, \{v_i\}, \{v_i^*\}); \\ D_{w_1}^\varphi &= M(x_1^\varphi, \{\omega_i\}, \{u_i^*\}) = D_{w_1}^0(1-Z), \quad D_{w_2}^\varphi = M(x_2^\varphi, \{\omega_i\}, \{v_i^*\}) = D_{w_2}^0(1+Z). \end{aligned}$$

Let  $R = M(id, \{\check{\alpha}_i\}, \{\omega_i\})$  and define

$$M_1 = (1-Z)^{-1}ZR({}^tD_{w_1}^\varphi), \quad M_2 = D_{w_2}^\varphi ZR(1+{}^tZ)^{-1},$$

now it is a straightforward computation to verify that these two matrices satisfy the required properties.

Let now  $d$  be the the map corresponding to  $\Delta$  with respect to the bases  $\{u_i, v_j, \omega_r\}$  and  $\{u_i^*, v_j^*, \check{\alpha}_r\}$ . We want to show that the image of  $d$  is a split direct summand, and to calculate the rank of  $\Delta$ . The result will then follow from the proposition on page 34 in [D-P1], due to restrictions imposed to  $l$ . From the results in [D-K-P2] we now that the map corresponding to  $(C_{w_1}^0, D_{w_1}^\varphi) : V_{w_1} \oplus \text{SP} \longrightarrow V_{w_1}$  is surjective with kernel

$$\{(u_{(1-\varphi)\lambda}, (1+(1-\varphi)^{-1}w_1(1-\varphi))\lambda) \mid \lambda \in \text{SP}\}.$$

Similarly the map corresponding to  $(-C_{w_2}^0, D_{w_2}^\varphi) : V_{w_2} \oplus \text{SP} \longrightarrow V_{w_2}$  is surjective with kernel

$$\{(-v_{(1+\varphi)\lambda}, (1+(1+\varphi)^{-1}w_2(1+\varphi))\lambda) \mid \lambda \in \text{SP}\}.$$

Here, for  $\lambda = \omega_r$  ( $r = 1, \dots, n$ ), we have defined

$$I_r^1 = \{k \in \{1, \dots, t\} \mid i_k = r\}, \quad I_r^2 = \{k \in \{1, \dots, m\} \mid j_k = r\}; \quad u_\lambda = \sum_{k \in I_r^1} u_k, \quad v_\lambda = \sum_{k \in I_r^2} v_k$$

and the extension of the definition of  $u_\lambda, v_\lambda$  to all  $\lambda \in P$  is the only compatible with linearity. We start the study of  $d$ . We consider the first row  $(C_{w_1}^0, 0, D_{w_1}^\varphi) : V_{w_1} \oplus V_{w_2} \text{SP} \longrightarrow V_{w_1}$ ; it is surjective with kernel

$$H = \{(u_{(1-\varphi)\lambda}, v, (1 + (1 - \varphi)^{-1}w_1(1 - \varphi))\lambda) \mid \lambda \in \text{SP} \ v \in V_{w_2}\}.$$

Our aim is now to study the image of the restriction of  $d$  on  $H$ . We proceed as follows. We define the composite map  $f : V_{w_2} \oplus \text{SP} \longrightarrow H \longrightarrow V_{w_2}^* \oplus \text{SQ}$ ,

$$(v, \lambda) \mapsto (u_{(1-\varphi)\lambda}, v, (1 + (1 - \varphi)^{-1}w_1(1 - \varphi))\lambda) \mapsto d(u_{(1-\varphi)\lambda}, v, (1 + (1 - \varphi)^{-1}w_1(1 - \varphi))\lambda).$$

With respect to the bases  $\{v_i, \omega_j\}, \{v_i^*, \check{\alpha}_j\}$ ,  $f$  is represented by the matrix

$$\begin{pmatrix} -C_{w_2}^0 & D_{w_2}^\varphi(1 + (1 + \Phi)^{-1}W_1(1 - \Phi)) \\ -{}^t D_{w_2}^\varphi & -(1 + \Phi)(1 - W_1)(1 - \Phi) \end{pmatrix},$$

where  $W_1, \Phi$  are the matrices representing  $w_1, \varphi$  respectively with respect to the basis  $\{\omega_j\}$ . To study this matrix is equivalent to study its opposite transpose (since we are essentially interested in their elementary divisors). Let  $M = (1 + (1 - \Phi)^{-1}W_1(1 - \Phi))$ ,  $N = (1 + \Phi)(1 - W_1)(1 - \Phi)$ . We consider therefore the matrix

$$\begin{pmatrix} -C_{w_2}^0 & D_{w_2}^\varphi \\ -{}^t M {}^t D_{w_2}^\varphi & {}^t N \end{pmatrix}.$$

Let  $g : V_{w_2} \oplus \text{SP} \longrightarrow V_{w_2}^* \oplus \text{SQ}$  be the map represented by this matrix with respect to the bases  $\{v_i, \omega_j\}, \{v_i^*, \check{\alpha}_j\}$ . Then  $(-C_{w_2}^0, D_{w_2}^\varphi) : V_{w_2} \oplus \text{SP} \longrightarrow V_{w_2}^*$  is surjective with kernel

$$L = \{(-v_{(1+\varphi)\lambda}, (1 + (1 + \varphi)^{-1}w_2(1 + \varphi))\lambda) \mid \lambda \in \text{SP}\}$$

and we are left to study the following composite  $e : \text{SP} \longrightarrow L \longrightarrow \text{SQ}$ ,

$$\lambda \mapsto (-v_{(1+\varphi)\lambda}, (1 + (1 + \varphi)^{-1}w_2(1 + \varphi))\lambda) \mapsto g((-v_{(1+\varphi)\lambda}, (1 + (1 + \varphi)^{-1}w_2(1 + \varphi))\lambda).$$

With respect to the bases  $\{\omega_i\}$  and  $\{\check{\alpha}_i\}$ ,  $e$  is represented by the matrix

$${}^t M(1 - \Phi)(1 - W_2)(1 + \Phi) + {}^t N(1 + \Phi)^{-1}W_2(1 + \Phi),$$

that is

$$(1 + {}^t(1 - \Phi){}^t W_1 {}^t(1 - \Phi)^{-1})(1 - \Phi)(1 - W_2)(1 + \Phi) + {}^t(1 - \Phi)(1 - {}^t W_1) {}^t(1 + \Phi)(1 + (1 + \Phi)^{-1}W_2(1 + \Phi)).$$

Since  $w$  is an isometry and  $\varphi$  is skew (one should use at each step appropriate bases), we get that  $e(\lambda)$  is the element

$$((1 + (1 + \varphi)w_1^{-1}(1 + \varphi)^{-1})(1 - \varphi)(1 - w_2)(1 + \varphi) + (1 + \varphi)(1 - w_1^{-1})(1 - \varphi)(1 + (1 + \varphi)^{-1}w_2(1 + \varphi))\lambda,$$

that is

$$e(\lambda) = (1 + (1 + \varphi)w_1^{-1}(1 + \varphi)^{-1})(1 - \varphi)(1 - w_2)(1 + \varphi) + (1 + \varphi)(1 - w_2^{-1})(1 - \varphi)(1 + (1 + \varphi)^{-1}w_2(1 + \varphi)).$$

It follows that

$$e(\lambda) = 2(1+\varphi)(1-\varphi) - 2(1+\varphi)w_1^{-1}(1+\varphi)^{-1}(1-\varphi)w_2(1+\varphi) = 2(1+\varphi)(1-w_1^{-1}\vartheta^{-1}w_2\vartheta)(1-\varphi).$$

Since both SP and SQ are invariant under  $2(1-\varphi)$  and  $2(1+\varphi)$ , we are left to study the map  $1-w_1^{-1}\vartheta^{-1}w_2\vartheta : \text{SP} \rightarrow \text{SQ}$ . The restriction imposed to  $l$  imply that the image of  $1-w_1^{-1}\vartheta^{-1}w_2\vartheta$  is a split direct summand.

It is also clear at this point that the rank of  $\Delta$  is precisely  $l(w_1) + l(w_2) + rk(1-w_1^{-1}\vartheta^{-1}w_2\vartheta)$ . But  $rk(1-w_1^{-1}\vartheta^{-1}w_2\vartheta) = rk(\vartheta w_1 - w_2\vartheta) = rk((1+\varphi)w_1(1-\varphi) - (1-\varphi)w_2(1+\varphi))$  and we are done.  $\square$

**5.7. Corollary** *Let  $l$  be a  $\varphi$ -good integer and let  $p$  be a point of the symplectic leaf  $\Theta$  of  $G$ . Then the dimension of any representation of  $F_\varepsilon^\varphi[G]$  lying over  $p$  is divisible by  $l^{\frac{1}{2}\dim\Theta}$ .*

**5.8.** As a consequence of (2.2) we have the following isomorphisms of Hopf algebras

$$R_q^\varphi[B_+] \simeq \Gamma^\varphi(\mathfrak{b}_-)_{op}, \quad R_q^0[B_+] \simeq \Gamma^0(\mathfrak{b}_-)_{op}.$$

Now the algebra  $\Gamma^\varphi(\mathfrak{b}_-)$  is equal to the algebra  $\Gamma^0(\mathfrak{b}_-)$  and so we have the isomorphism of algebras

$$R_q^\varphi[B_+] \simeq R_q^0[B_+]$$

and similarly for the case  $B_-$ . Then the representations of  $F_\varepsilon^\varphi[G]$  over the sets  $X_{(w,1)}$  and  $X_{(1,w)}$  are like in the case  $\varphi = 0$  (they are studied in [D-P2]). In particular there is an isomorphism between the one dimensional representations of  $F_\varepsilon^\varphi[G]$  and the points of the Cartan torus  $T$  (given esplicitely [D-L]).

## Appendix

In **4.** we determined the dimension  $d_\varphi(w_1, w_2)$  of a symplectic leaf  $\Theta$  contained in  $X_{(w_1, w_2)}$ ;

$$d_\varphi(w_1, w_2) = l(w_1) + l(w_2) + rk((1+\varphi)w_1(1-\varphi) - (1-\varphi)w_2(1+\varphi)).$$

This means, of course, that  $d_\varphi(w_1, w_2)$  is an even integer. Here we give a direct proof of this fact in the more general context of finite Coxeter groups. Using the definitions from [H], let  $W = \langle s_1, \dots, s_n \rangle$  be a finite Coxeter group of rank  $n$ ,  $\sigma : W \leftarrow GL(V)$ , the geometric representation of  $W$ ,  $B$  the  $W$ -invariant scalar product on  $V$ ,  $\Phi$  the root system of  $W$ ,  $l(\cdot)$  the usual length function on  $W$ . We recall a fact proved in [C] for Weyl groups and which holds with the same proof for finite Coxeter groups. Each element  $w$  of  $W$  can be expressed in the form  $w = s_{r_1} \cdots s_{r_k}$ ,  $r_i \in \Phi$ , where  $s_v$  is the reflection relative to  $v$ , if  $v$  is any non zero element of  $V$ . Denote by  $\bar{l}(w)$  the smallest value of  $k$  in any such expression for  $w$ . We get

**Lemma**  $\bar{l}(w) = rk(1-w)$ .

*Proof.* It is Lemma 2 in [C].  $\square$

We can now prove

**Proposition 1** *Let  $w_1, w_2$  be in  $W$ . Then  $l(w_1) + l(w_2) + rk(w_1 - w_2)$  is even.*

*Proof.* We have

$$rk(w_1 - w_2) = rk(1 - w_2 w_1^{-1}) = \bar{l}(w_2 w_1^{-1}) \equiv l(w_2 w_1^{-1}) \mod 2.$$

But  $l(w_2 w_1^{-1}) \equiv l(w_2) + l(w_1^{-1}) \mod 2$  and finally  $l(w_1^{-1}) = l(w_1)$ . Hence  $l(w_1) + l(w_2) + rk(w_1 - w_2) \equiv l(w_1) + l(w_2) + l(w_2) + l(w_1) \equiv 0 \mod 2$ .  $\square$

Suppose now  $\varphi$  is an endomorphism of  $V$  which is skew relative to  $B$ , and let  $\vartheta$  be the isometry  $(1 + \varphi)^{-1}(1 - \varphi)$ . To prove the general result, we recall that, if  $\eta$  is an isometry of  $V$  and  $r$  is the rank of  $1 - \eta$ , then  $\eta$  can be written as a product of  $r + 2$  reflections (cf. [S], where a more precise statement is given). In particular if  $\eta = s_{v_1} \cdots s_{v_k}$ , then  $rk(1 - \eta) \equiv k \mod 2$ .

**Proposition 2** *Let  $w_1, w_2$  be in  $W$ . Then  $l(w_1) + l(w_2) + rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi))$  is even.*

*Proof.* We have  $rk((1 + \varphi)w_1(1 - \varphi) - (1 - \varphi)w_2(1 + \varphi)) = rk(1 - \vartheta w_2 \vartheta^{-1} w_1^{-1})$ . If we write  $\vartheta, w_1, w_2$  as products of  $a, a_1, a_2$  reflections respectively, we get from the previous observation that  $rk(1 - \vartheta w_2 \vartheta^{-1} w_1^{-1}) \equiv a + a_2 + a + a_1 \mod 2$ . Hence

$$rk(1 - \vartheta w_2 \vartheta^{-1} w_1^{-1}) \equiv rk(1 - w_2 w_1^{-1}) \equiv rk(w_1 - w_2) \mod 2$$

and the result comes from prop.1.  $\square$

### Acknowledgements

The authors are grateful to C. De Concini for helpful discussions and enlightening explanations.

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